Computation of Residue When a Function $f(z)$ Has a Pole of Order n

We are looking to compute

(1) $$\oint_C f(z)dz$$

when $f(z)$ is not analytic at a point z_0 and C includes the point z_0. In this case, we write the Laurent series of $f(z)$ around z_0 to get

(2) $$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \ .$$

Substituting (2) in (1), we get

(3) $$\oint_C f(z)dz = \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_0)^k \, dz = a_{-1} \cdot 2\pi i \ .$$

The last equality follows as

(4) $$\oint_C (z - z_0)^k \, dz = \begin{cases} 0 & if \;\; k \neq -1 \\ 2\pi i & if \;\; k = -1 \end{cases}.$$

When the function $f(z)$ has a pole of order n at z_0, its Laurent series expansion looks like

(5) $$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1 (z - z_0) + \ldots$$

We need to compute a_{-1} from (5) in order to evaluate the integral in (3). Multiply both sides of (5) by $(z - z_0)^n$ to get

(6) $$\begin{aligned}(z - z_0)^n f(z) = {} & a_{-n} + a_{-n+1} (z - z_0) + a_{-n+2} (z - z_0)^2 + \ldots \\ & + a_{-2} (z - z_0)^{n-2} + a_{-1} (z - z_0)^{n-1} + a_0 (z - z_0)^n + a_1 (z - z_0)^{n+1} + \ldots\end{aligned}$$

The RHS is a sum of terms, each term being a polynomial. Differentiating both sides wrt z $(n - 1)$ times and taking the limit as $z \to z_0$, we get

(7) $$\lim_{z \to z_0} \frac{d^{\,n-1}}{dz^{\,n-1}} (z - z_0)^n \, f(z) = (n-1)! \, a_{-1} \, .$$

Dividing both sides of (7) by $(n - 1)!$, we get a_{-1}, the residue of $f(z)$ at $z = z_0$. Replacing a_{-1} in (3) by this expression, we have the result,

(8) $$\oint_C f(z)dz = 2\pi i \cdot \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{\,n-1}}{dz^{\,n-1}} (z - z_0)^n \, f(z) = 2\pi i \cdot residue \, of \;\; f(z) \, at \; z = z_0 \, .$$