NUS ECE Department EE2012 Analytical Methods in ECE Mid-term Test

Time Allowed: 50++ min Max Marks: 120

Answer all questions. All questions carry equal marks. Mark your answers on this sheet. Circle the appropriate letter for the True/False type choices.

Question 1. A bio-experiment consists of a sequence of SI trials. Each trial has 3 outcomes H, T and E. At the i-th trial, $i = 1,2,\ldots,$ $P(H) = 0.2$, $P(T) = 0.3 + 1/(10i)$ and $P(E) = 0.5 - 1/(10i)$.

A) The trials can be classified as Bernoulli trials. **T or F**
Bernoulli trials have only two possible outcomes. We have three possible outcomes here.
B) The trials can be termed as Bernoulli trials if we are interested only in the occurrence of H.
T or F
If we are only interested in H, we have only two outcomes H and 'Not H'. The successive trials are SI and the prob(H) = 0.2 and prob(Not H) = 0.8. All three conditions of Bernoulli trials are satisfied.
C) The trials can be termed as Bernoulli trials if we are interested only in the occurrence of T.
T or F
Here we only have two possible outcomes T and 'Not T'. However the probability at each trial is not constant.
D) If 100 trials are performed, write an expression for the probability of getting 50 H.
Answer: We use the binomial pdf with $N = 100$ and $p = 0.2$.

$$P(50H) = \frac{100!}{50!50!} 0.2^{50} 0.8^{50}$$

Question 2. Let X_1, X_2, … be a sequence of Gaussian RVs; Y_1, Y_2, … a sequence of binomial RVs with $N = 10$ and $p = 0.2$; and Z_1, Z_2, … a sequence of binomial RVs with $N = 20$ and $p = 0.3$. All RVs are SI.

A) $X_1 + X_2 + \ldots + X_M$ is Gaussian only when M is large. **T or F**
The sum will always give a Gaussian rv as the original rvs $X_1, X_2, \ldots, X_M$ are Gaussian. M can be small or large.
B) $(Y_1 + Z_1) + (Y_2 + Z_2) + \ldots + (Y_M + Z_M)$ is binomial. **T or F**
$(Y_1 + Z_1)$ is not a binomial rv. The sum is not either.
C) $Z_1 + Z_2 + \ldots + Z_M$ is Gaussian only when M is large. **T or F**
This happens by the statement of the central limit theorem.
D) $\sum_{i=1}^{M} (X_i + Y_i + Z_i)^{2047}$ is Gaussian when M is large. **T or F**
Since all the rvs are SI, the rvs $(X_1 + Y_1 + Z_1)^{2047}$, $(X_2 + Y_2 + Z_3)^{2047}$, … are SI. The statement follows from central limit theorem.
Question 3. Let $h(x)$ and $g(y)$ be marginal pdf of X and Y, respectively, and $f(x, y)$ be the joint pdf of (X, Y).

A) $E(XY) = E(X)\,E(Y)$ implies $f(x, y) = g(x)h(y)$. **T or F**

Expectations are average properties of pdf and do not say anything about the pdfs themselves.
B) If X and Y are uncorrelated Gaussian RVs then $f(x, y) = g(x)h(y)$. **T or F**
In general, if two rvs are uncorrelated, it does not mean that they are SI, but it happens for Gaussian rvs which is the case here. However, here the role of the pdfs are reversed in the rhs of $f(x, y) = g(x)h(y)$. That makes the answer F. If the statement were $f(x, y) = g(y)h(x)$, the answer will be T.
C) If X and Y are SI then MGF($X + Y$) = MGF(X) MGF(Y). **T or F**
MGF($X + Y$) = E[$e^{s(X+Y)}$] = E[e^{sX}] E[e^{sY}] = MGF(X) MGF(Y).
D) If X and Y are SI with zero mean, then they must be orthogonal. **T or F**
SI implies E[XY] = E[X] E[Y] = 0. Hence the rvs are orthogonal.

Question 4. Let $X \sim N(0.1, 0.05)$ and $Y \sim N(0.2, 0.05)$. In addition, cov(X, Y) = 0.03.

A) $V(X + Y) = 0.1$. **T or F**
$V(X + Y) = E[\{(X + Y) - (m_x + m_y)\}^2]$
$= E[(X - m_x)^2] + E[(Y - m_y)^2] + 2\,E[(X - m_x)(Y - m_y)] = 0.05 + 0.05 + 2 \times 0.03 = 0.16$.
Note: Variances do not add up if he rvs are not SI.
B) $V(X + Y) > 0.2$. **T or F**
See the analysis for part A.
C) $P(X + Y < 0.3) = 0.2$. **T or F**
$X + Y$ is Gaussian with mean 0.3. Hence $P(X + Y < 0.3) = 0.5$.
D) $E(X + Y) = 0.3$. **T or F**
$E(X + Y) = E(X) + E(Y) = 0.3$.

Question 5.
A) A sample of size one is a biased estimator of the population mean. **T or F**
Regardless of the size of the sample, the average is an unbiased estimator of the sample mean.
B) An unbiased estimator is the most efficient estimator. **T or F**
No, we can have lots of unbiased estimators. The most efficient estimator will be the one that minimizes the variance of the estimate. It has nothing to do with whether it is biased or not.
C) An efficient estimator is also a consistent estimator. **T or F**
Not necessarily. The variance may not go to zero as the sample size gets large.
D) ML estimator is always an unbiased estimator. **T or F**
Not necessarily. Once the ML estimator is derived, we need to test if it is an unbiased estimator. We have seen examples when the ML estimator was biased.

Question 6.
A) $P(A \mid B)P(B) = 0$ if A and B are mutually exclusive events. **T or F**
$P(A|B)\,P(B) = P(AB) = 0$ if A and B are mutually exclusive.
B) $P(A \mid B) = P(B \mid A)$ if A and B are SI events. **T or F**
If A and B are SI events then $P(A \mid B) = P(B \mid A)$ implies that $P(A) = P(B)$ which is not true in general.
C) $P(A \cup B) - P(A) - P(B) = 0$ if A and B are SI events. **T or F**
If A and B are SI events then $P(A \cup B) - P(A) - P(B) = -P(AB) = -P(A)P(B) \neq 0$ in general.
D) $P(A \mid B)P(B) = P(A \cap B)$ if A and B are SI events. **T or F**

This is always true even when the events are not SI.

Question 7. Let $g(x)$ and $h(y)$ be the pdfs of RVs X and Y, respectively, and $f(x, y)$ be their joint pdf. Let A be a constant.

A) $f(x, y) = Ax^2y^3$, $0 < x < 1, 0 < y < x \Rightarrow X$ and Y are SI. **T or F**

The region over which the rv (X, Y) is defined is not a rectangle.

B) $f(x, y) = Ae^{-(x^2+y^2)}$, $0 < y < 1, 0 < x < 1 \Rightarrow X$ and Y are SI. **T or F**

The function $f(x, y)$ can be written as a product of a function of X and a function of Y and the region is a rectangular one.

C) $f(x, y) = A\ln(x + y)$, $0 < x < 1$, $0 < y < 1$ $\Rightarrow X$ and Y are SI. **T or F**

The function $f(x, y)$ cannot be written as a product of a function of X and a function of Y though the region is a rectangular one.

D) If X and Y are uncorrelated then $f(x, y) = g(x)\, h(y)$. **T or F**

If two rvs are uncorrelated, it says nothing about how the pdfs behave.

Question 8. In a telecommunication network, the transmitted signal S and received signal R are related as $R = S + N$, where N is the noise in the channel modelled as a standard normal RV. It is given that S is either 10 Volt or −10 Volt, and both are transmitted with equal probability.

A) $E(S) = 0$. **T or F**

$E(S)$ = 10 x 0.5 + (-10) x 0.5 = 0.

B) The mean of R given that S = 10 is the same as mean of R given that S = −10. **T or F**

Given S = 10, R is a Guassian rv with mean 10 and variance 10. Thus $E(R \mid S = 10) = 10$.
Similarly, $E(R \mid S = -10) = -10$.

C) By central limit theorem, pdf of R is approximately Gaussian. **T or F**

R is only sum of two rvs S and N. S is not Gaussian. Central limit theorem does not apply.

D) Sketch the pdf of R.

$f(r) = f(r \mid S = 10)\, p(S = 10) + f(r \mid S = -10)\, p(S = -10)$
$= 0.5\, \{f(r \mid S = 10) + f(r \mid S = -10)\}$.

Finally, $f(r \mid S = 10) \sim N(10, 1)$ and $f(r \mid S = -10) \sim N(-10, 1)$. A sketch is given below.