Name: ___________________________

NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical & Computer Engineering
EE2003 Engineering Mathematics 2B

Sample Mid-term Test Questions and Solutions

Max Marks 100 **Time Allowed 55 min**

1. Show that if variance of a rv $X = 0$, then it takes only one value with probability 1. Identify that value. (30 marks)

$$V(X) = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx.$$

Solution:

Since, $(x - m)^2$ and $f(x)$ are always positive, $V(X)$ is always positive. The only way we could have a 0 value is when $f(x) = \delta(x - m)$. In other words, X is a rv that can take only one value equal to m (also = mean = m = E(X)). This is a trivial case but helps us to design our systems (receivers for communication systems, for example) to drive the probability to 1 in order to improve their reliability.

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2. For this part assume that if variance of a rv $X = 0.001$, then variance is 0. We are given a bag containing s black and $N - s$ white balls. s is unknown and N is known. Also, it is known that N does not exceed 10,000. We are allowed to draw one ball at a time, observe its color and replace it back. We are allowed to repeat this procedure as many times as we wish. Develop a method to determine s with probability 1. State clearly any assumptions made and the pdfs of the underlying random variables. Also describe the usefulness of the CLT (Central limit theorem) for this problem.

(35 marks)

Solution:

We draw n balls **with replacement** and note the number of times a black ball shows up. Let this be f. Then we say (from the relative frequency interpretation of probability),

$$f / n = s / N$$

provided n is large enough. This can now be further extended to say that

$$s = N \,.\, (f / n).$$

For it to work perfectly, we must have variance of RHS = 0.

Variance of RHS = $(N / n)^2$ Var(f).

Since f is binomial (experiment is done with replacement) with parameters n and (s / N),

$$\text{Variance of RHS} = (N / n)^2 \,.\, n \,.\, (s / N) \,.\, (1 - s / N)$$
$$= (N^2 / n) \,.\, (s / N) \,.\, (1 - s / N) <= (N^2 / n) \,.\, (1 / 4).$$

For variance to go to 0 or to 0.001 as stated,

$$(N^2 / n) \,.\, (1 / 4) < 0.001 \text{ or } 250 \,.\, N^2 < n.$$

The CLT can be used further to arrive at a range of values that will be occupied by s with high probability (approaching 1). However, the problem as stated does not need the approximation that CLT deals with.

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3. Consider a **three-face** coin. The faces and their probabilities of occurrence at any given toss are given by

   Success: probability = r
   Failure: probability = s
   Don't care: probability = t

   It is clear that $r + s + t = 1$. This coin is tossed N times. Define a two-dimensional rv as

   (X, Y) = (total number of successes, total number of failures).

   (a) Are these Bernoulli trials?
   (b) Write the 2-dimensional joint pdf of (X, Y).
   (c) Write the pdf of the random variable Z defined as $Z = X + Y$.
   (d) Write the marginal pdf of X.
   (e) Write the conditional pdf of X given that $Y = 5$.
   (f) Are X and Y SI?

   State clearly any assumptions made. (35 marks)

   Solution:

   (a) No. They require only two possible outcomes for each trial. We have three.
   (b) $P(X = a, Y = b) = \dfrac{N!}{a!b!(N - a - b)!} r^a s^b t^{N-a-b}$.
   This is defined for all values of a and b such that $a \geq 0$, $b \geq 0$ and $a + b \leq N$. The derivation can be done along the same lines as the derivation for binomial pdf.
   (c) $Z = X + Y$ means we have either (success or failure) as one outcome or don't care as the other outcome. P(success or failure in one trial) = $r + s$. Thus Z is binomial with parameters N and $r + s$.
   (d) It is clear that $X = a$ means that we are looking for a successes in N trials. This is binomial pdf with parameters N and r.
   (e) Given that $Y = 5$, means we are now measuring successes in $N - 5$ trials. For $X = a$, we have a successes in $N - 5$ trials. This is binomial pdf with parameters $N - 5$ and r.
   (f) No. This is clear from the form of P($X = a$, $Y = b$) as stated in (b) above.