NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical Engineering
EE2003 Engineering Mathematics 2B

Sample Mid-Term Test Questions and Solutions

1. The pdf of a rv T, defined as T = duration of a phone call, is modeled as exponential pdf. Therefore,

   $f(t) = a \exp[-at]$, $t > 0$.

   Show that T satisfies the memory-less property given by

   $P(T \leq t + s \mid T > t) = P(T \leq s)$.

   Note: This pdf is used extensively to model the duration of phone calls in the telephone network.

   (30 Marks)

Solution:

$$P(T \leq t + s \mid T > t) = \frac{P(t < T \leq t+s)}{P(T > t)} = \frac{\int_t^{t+s} ae^{-at}\,dt}{\int_t^{\infty} ae^{-at}\,dt} = 1 - e^{-as} = P(T \leq s).$$

Marks: Each equal sign is worth 10 points.

Note: This can be interpreted in the following manner. If you have been on the phone for more than t sec, the probability that you will terminate the call in the next s sec is same as if you just started talking. In other words, you tend to forget the past behavior. This is the worst kind of assumption as in most cases humans do remember the duration and their probability of terminating the call goes up as the duration of the call goes up. No wonder the phone system works so well as it is designed with the worst kind of behavior in mind!

2. We are modeling arrivals (all arrivals are SI) in an interval $(t, t + dt]$ as a rv X having pdf $P(X = 0) = 1 - a \cdot dt$, $P(X = 1) = a \cdot dt$.

   (a) Find $E(X)$.
   (b) Find $V(X)$.
   (c) Write the pdf of the rv Y = # arrivals in $[0, T]$, $T = N \cdot dt$.
   (d) Find $E(Y)$ and $V(Y)$.
   (e) Can we approximate this pdf by a Gaussian pdf? Justify. If yes, find the parameters of this Gaussian pdf.
   (f) Is this pdf same as the Poisson pdf? State the conditions involved.

(40 Marks)

Solution:

(a) 5 points: $E(X) = 0 \,.\, (1 - a \,.\, dt) + 1 \,.\, (a \,.\, dt) = a \,.\, dt$.

(b) 5 points: $E(X^2) = 0 \,.\, (1 - a \,.\, dt) + 1 \,.\, a \,.\, dt = a \,.\, dt$

$$V(X) = E(X^2) - E^2(X) = a \,.\, dt \,.\, (1 - a \,.\, dt).$$

(c) 6 points: We can split up the interval $[0, T]$ in N non-overlapping parts. Let X_i denote the arrivals in the i-th interval. Each of X_i can take two values: 0 and 1 with probabilities $P(X_i = 0) = 1 - a \,.\, dt$, $P(X_i = 1) = a \,.\, dt$. It is clear that $Y = X_1 + X_2 + \ldots + X_N$, where these rvs are SI. Thus Y has binomial pdf with parameters N and $p = a \,.\, dt$.

(d) 8 points: Since Y is a sum of N rvs,
   $E(Y)$ = sum of the expected values or $E(Y) = N \,.\, a \,.\, dt$.
   Since Y is a sum of N SI rvs,
   $V(Y)$ = sum of the variances $= N \,.\, a \,.\, dt \,.\, (1 - a \,.\, dt)$.

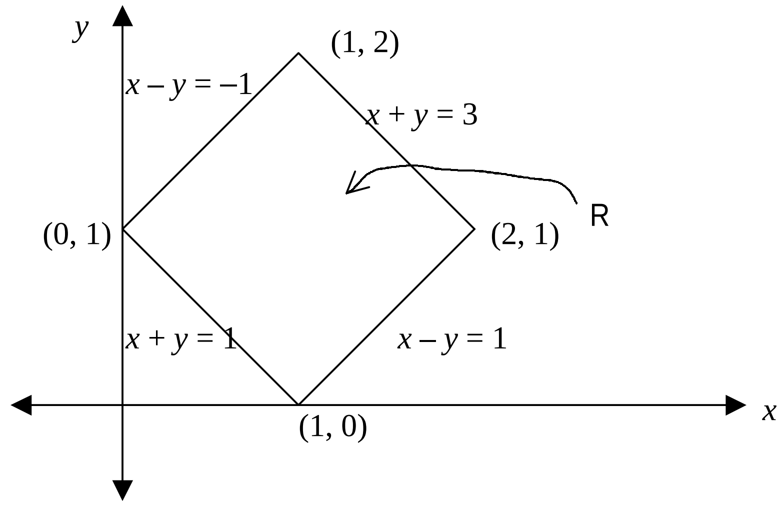
(e) 8 points: Yes as long as N is large. Recall that the two conditions for Central Limit Theorem (N large, Y is a sum of SI rvs) are satisfied. The parameters are the mean and variance that remain the same as in (d).

(f) 8 points: For Poisson, N must tend to infinity and dt to 0 in a way that their product $T = N \,.\, dt$ is finite. If these conditions are satisfied, then the resulting rv will have Poisson pdf with parameter $a \,.\, T$.

3. Consider the joint pdf of 2-dimensional rv (X, Y) given by

$f(x, y) = k$,

over the region R sketched below.



(a) Find k.
(b) Define two rvs as $A = X + Y$ and $B = X - Y$. Mark the region corresponding to the event $a < A \le a + da$ and $b < B \le b + db$, for (a, b) belonging to R. Hence evaluate its probability.
(c) Are A and B SI. Justify.

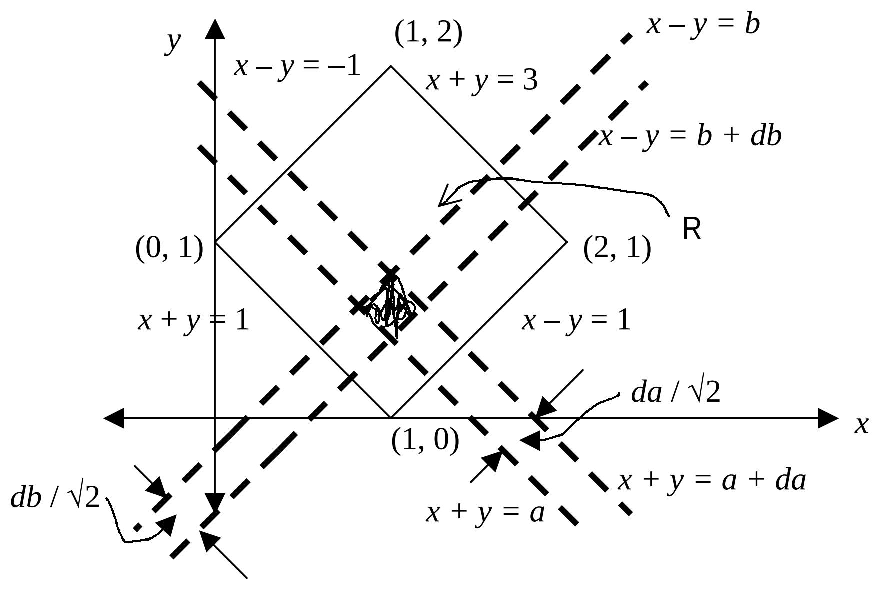
(30 Marks)

Solution:

(a) 8 points: k is evaluated by integrating $f(x, y)$ over R and putting it equal to 1. Thus,

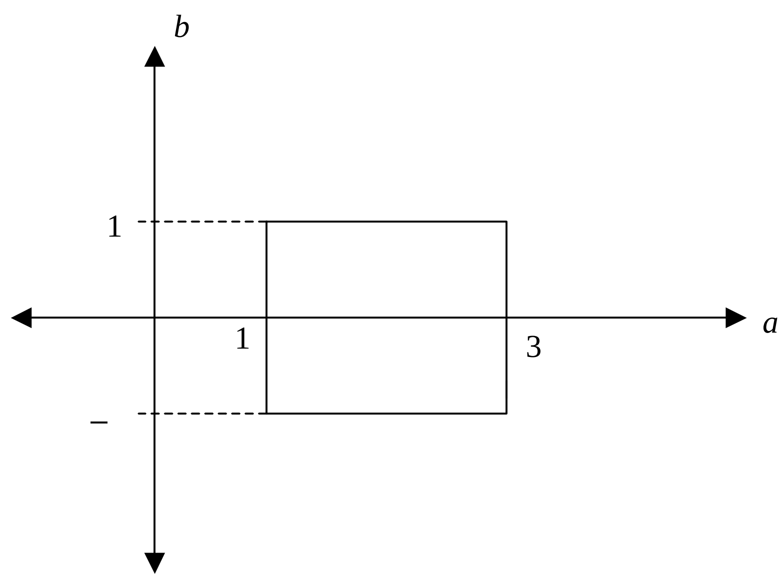
$$\iint_{R} f(x,y)dxdy = 1 \Rightarrow \iint_{R} kdxdy = 1 \Rightarrow k\iint_{R} dxdy = 1 \Rightarrow k \cdot \text{area of R} = 1 \Rightarrow k = 1/2 .$$

(b) 8 points: The region corresponding to the event $a < A \le a + da$ and $b < B \le b + db$, for (a, b) belonging to R is shaded below.



6 point: The probability of this event is k . area of the infinitesimally small rectangle = (1/2) . ($da/\sqrt{2}$) . ($db/\sqrt{2}$) = (1/4) . da . db.

(c) 8 points: Since
$P(a < A \le a + da \text{ and } b < B \le b + db) = (1/4) \cdot da \cdot db$,
The joint pdf of (A, B) is given by $h(a, b) = 1/4$. For any value of A, B goes from –1 to 1. Similarly, for any value of B, A goes from 1 to 3. The region for (A, B) is sketched below:



Since the joint pdf of (A, B) is product of a function of a and a function of b, and the region for (A, B) is a rectangle with sides parallel to the axes, A and B are SI.