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NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical & Computer Engineering
EE2003 Engineering Mathematics 2B

Sample Mid-term Test Questions and Solutions

Max Marks 100 **Time Allowed 55 min**

1. We have a person who tosses a coin continuously (P(head) = r). We also have a machine that records the number of tosses. However it malfunctions and records the actual toss with probability p. The person tosses until 5000 tosses were recorded. Define a rv X as X = number of **actual** tosses; and a rv Y as Y = number of **actual** heads.

   (a) Write the pdf of X.
   (b) Write an expression for pdf of Y.
   (c) Write the conditional pdf of Y given that X = 6000.

(30 Marks)

Solution:

(a) We have recorded 5000 tosses. Therefore, the number of actual tosses, X, can take values 5000, 5001, … For P($X = k$), we have k tosses conducted and 5000 recorded. Therefore,

P($X = k$) = (k-th toss recorded and 4999 tosses recorded in the first k – 1 tosses)

$$= p \,.\, {}^{k-1}C_{4999}\, p^{4999}(1-p)^{k-1-4999} = {}^{k-1}C_{4999}\, p^{5000}(1-p)^{k-5000}\ ,\ k = 5000, 5001, \ldots$$

(b) $P(Y = y \mid X = k) = {}^{k}C_y r^y (1-r)^{k-y}$

Therefore, $P(Y = y) = \sum_{k=5000}^{\infty} P(Y = y . X = k) = \sum_{k=5000}^{\infty} P(Y = y | X = k) P(X = k)$.

Expressions for $P(Y = y \mid X = k)$ and $P(X = k)$ can now be substituted.

(c) This can be obtained by setting k = 6000 in $P(Y = y \mid X = k) = {}^{k}C_y r^y (1-r)^{k-y}$. We note that this is a binomial pdf with parameters 6000 and r.

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2. Consider the arrival of people at a fast food joint called Prodonalds. We are modeling arrivals in Prodonalds in an interval $(t, t + dt]$ as a rv X in the following manner

   $P(X = 0) = 1 - a \cdot dt$,
   $P(X = 1) = a \cdot dt$. There can't be more than 1 arrival in this interval.

   A person walking in can join two queues Q1 (happy folks) or Q2 (sad folks) depending on his mood. It is known that P(happy) = p while P(sad) = $1 - p$.

   (a) Find $E(X)$.
   (b) Find $V(X)$.
   (c) Write the pdf of the rv Y = # arrivals in Q1 in $(t, t + dt]$.
   (d) Find $E(Y)$ and $V(Y)$.
   (e) Hence or otherwise, write the pdf of a rv Z defined as Z = # arrivals in Q1 in (0, 500]. Is this pdf same as the Poisson pdf? Gaussian pdf? State the conditions involved.

   Make and state all assumptions clearly.

(40 Marks)

Solution:

(a) $E(X) = 0 \,.\, (1 - a \,.\, dt) + 1 \,.\, a \,.\, dt = a \,.\, dt$.

(b) $V(X) = \{0^2 \,.\, (1 - a \,.\, dt) + 1^2 \,.\, a \,.\, dt\} - E^2(X) = a \,.\, dt - a^2\, (dt)^2$.

(c) Y can take two values, 0 or 1.
P(Y = 0) = P(no arrival in Prod OR (arrival joins Q2 AND 1 arrival in Prod))
= P(no arrival in Prod) + P(arrival joins Q2 AND 1 arrival in Prod))
= $1 - a \,.\, dt$ + P(arrival joins Q2 | 1 arrival in Prod) . P(1 arrival in Prod)
= $1 - a \,.\, dt + (1 - p) \,.\, a \,.\, dt = 1 - p \,.\, a \,.\, dt$.
Perhaps a simpler argument is
P(Y = 1) = P(arrival joins Q1 AND 1 arrival in Prod)
= P(arrival joins Q1 | 1 arrival in Prod) . P(1 arrival in Prod)
= $p \,.\, a \,.\, dt$.

(d) Y is same as X with a replaced by $a \,.\, p$. Hence expressions (b) hold here with a replaced by $a \,.\, p$.

(e) This is same as Poisson pdf with parameter $a \,.\, p \,.\, 500$. This can also be approximated by Gaussian pdf as it is sum of large number of SI rvs each one of the type Y. The conditions are (i) all arrivals are SI, (ii) arrival rate remains the same throughout the time interval, and (iii) In an infinitesimally small interval, either 0 or 1 arrival can take place and nothing else.

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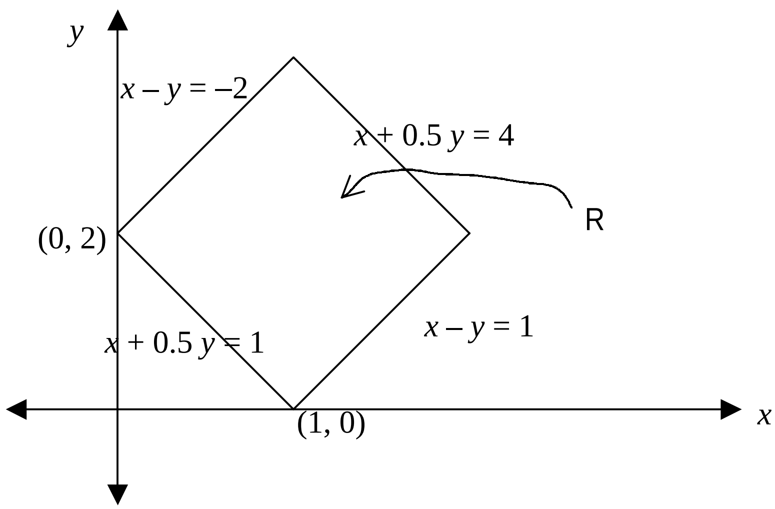
3. Consider the joint pdf of 2-dimensional rv (X, Y) given by

$f(x, y) = k,$

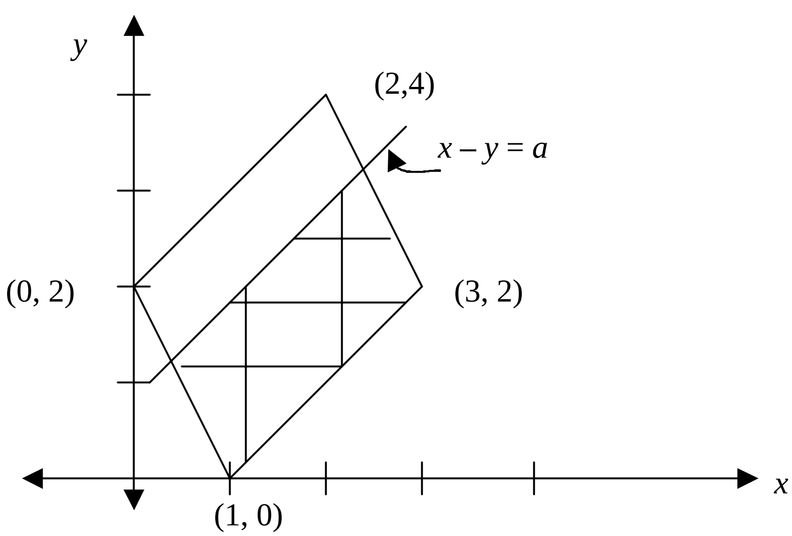
over the region R sketched below.



(a) Find k.
(b) Define a rv A as $A = X - Y$. Mark the region corresponding to the event $a < A$. Hence evaluate its probability.
(c) Obtain an expression for the marginal pdf of X.
(d) Are X and Y SI? Justify.

(30 Marks)

Solution:
(a) The figure redrawn with all the pertinent details is as follows:



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Since integral of $f(x, y)$ is 1 over the region;
$\int k\, dx\, dy = 1$ or $k \int dx\, dy = 1$ or k . area of the parallelogram = 1 or $k = 1/6$
as the area of the parallelogram is 6.

(b) The region is shaded. Its probability is
= k . area of the shaded part = (1/6) . 2 . (1 – a).

(c) For the marginal pdf of X,

$$f(x) = \int f(x, y)\,dy$$

$$f(x) = \int_{2-2x}^{x+2} (1/6)\,dy = \frac{1}{6}(3x),\ 0 < x <= 1$$

$$f(x) = \int_{x-1}^{x+2} (1/6)\,dy = \frac{1}{6}(3),\ 1 < x <= 2$$

$$f(x) = \int_{x-1}^{8-2x} (1/6)\,dy = \frac{1}{6}(9 - 3x),\ 2 < x <= 3.$$

(d) X and Y are not SI as the region over which they are defined is not a rectangular one with sides parallel to the X and Y axes.