NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical and Computer Engineering
EE2003 Engineering Mathematics 2B

Sample Mid Term Test Questions and Solutions

1. The pdf of a rv T, defined as T = duration of a phone call made by one person, is modeled as exponential pdf. Therefore,

   $f(t) = a \exp[-at]$, $t > 0$.

   A phone was used by 10 persons in a row. Let X be a rv denoting the duration of the total use of the phone. Find the pdf of X.

   Note: Exponential pdf is used extensively to model the duration of phone calls in the telephone network.

   (30 Marks)

   Solution:
   Let X_i be the rv denoting the i-th user. It is clear that $X = X_1 + \ldots + X_{10}$. Since the rvs $X_1, \ldots, X_{10}$ are SI, the pdf of X is the convolution of the individual pdfs. Using the Laplace transform,

   $F(s) = F_1(s) \,.\, F_2(s) \ldots F_{10}(s)$,

   where $F(s)$ is the Laplace transform of $f(x)$, the pdf of X, and $F_i(s)$ is the Laplace transform of $f_i(x)$, the pdf of X_i.

   $F(s) = a^{10} / (s + a)^{10}$.

   Taking inverse Laplace transform,

   $f(x) = a^{10} \,.\, \{x^9 / 9!\} \,.\, \exp[-ax]$, $x > 0$.

2. We are modeling arrivals (all arrivals are SI) in an interval $(t, t + dt]$ as a rv X having pdf $P(X = 0) = 1 - a \cdot dt$, $P(X = 1) = a \cdot dt$.

(a) Write the pdf of the rv Y = # arrivals in $(t, t + T]$, $T = N \cdot dt$.
(b) Find $E(Y)$ and $V(Y)$.
(c) Can we approximate the pdf of Y by a Gaussian pdf? Justify. If yes, find the parameters of this Gaussian pdf.
(d) Is this pdf same as the Poisson pdf? Justify. State the conditions and the parameters of the pdf involved.
(e) Using (d), Write an expression for

P(number of arrivals in$(t, t + T)$ is at most 30 | number of arrivals in $(t, t + T)$ is at least 10).

(40 Marks)

Solution:

(a) Y has binomial pdf with parameters N, and $p = a \cdot dt$.

(b) $E(Y) = N \,.\, p = N \,.\, a \cdot dt$, $V(Y) = N \,.\, p\,(1 - p) = N \,.\, (a \cdot dt) \,.\, (1 - a \cdot dt)$.

(c) Yes, we can if N is large. In this case, Y becomes Gaussian with mean and variance as given in (b).

(d) Yes, it is. As long as all arrivals are SI and $E(Y) = N \,.\, a \cdot dt = a \cdot T$, that is, $N \cdot dt = T$, a finite value. $P(Y = k) = e^{-at}(aT)^k / k!$.

(e) P(number of arrivals in$(t, t + T)$ is at most 30 | number of arrivals in $(t, t + T)$ is at least 10)

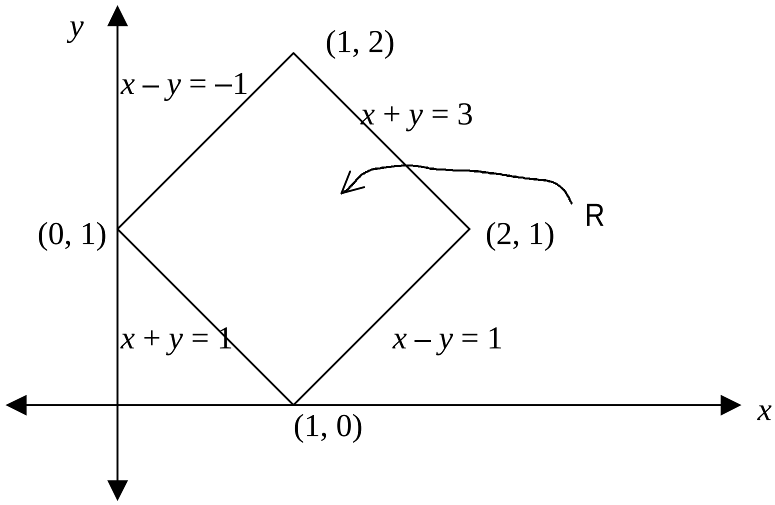
$= P(Y <= 30 \mid Y > = 10) = P(10 <= Y <= 30) / P(Y >= 10)$

$$= \frac{\sum_{k=10}^{30} e^{-aT} (aT)^k / k!}{\sum_{k=10}^{\infty} e^{-aT} (aT)^k / k!}.$$

3. Consider the joint pdf of 2-dimensional rv (X, Y) given by

$f(x, y) = k$,

over the region R sketched below.



(a) Find k.
(b) Find the marginal pdf of X.
(c) Define a rv as $A = X + Y$. Mark the region corresponding to the event $a < A \le a + da$, for a belonging to R. Hence evaluate $P(a < A \le a + da)$ and the pdf of A.
(d) Are X and Y SI? Justify.

(30 Marks)

Solution:

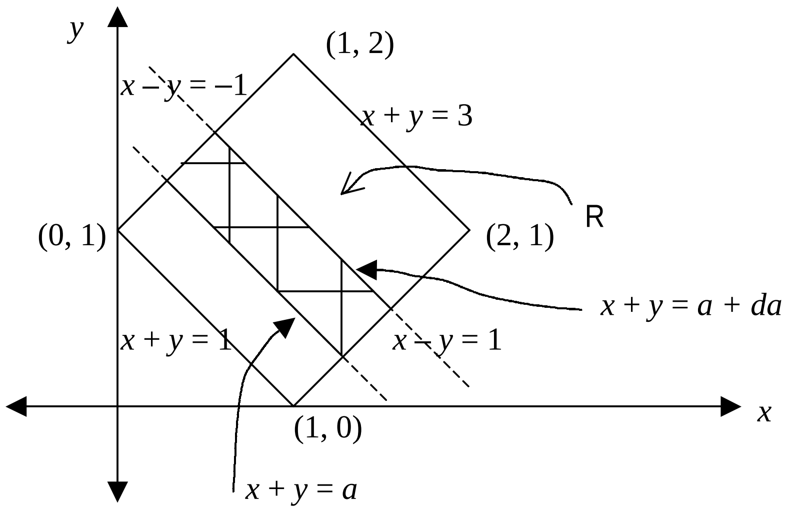
(a) Integral of $f(x, y) = 1$. Since $f(x, y) = k$, we have k . area of region = 1 or $k = 1 / 2$.

(b) $f(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} 0.5 dy$

$$= \int_{1-x}^{1+x} 0.5 dy = x, \quad 0 < x \le 1$$

$$= \int_{x-1}^{3-x} 0.5 dy = 2 - x, \quad 1 < x \le 2.$$

(c)



P($a < A \leq a + da$) = k . area of the marked region = 0.5 . da

By definition,

$f(a)$ = pdf of A = 0.5 for $1 < a <= 3$.

(d) No. X and Y are not SI as the region over which they are defined is not a rectangular one with its sides parallel to the x and y axes.