Student Matric Number:

National University of Singapore
Department of Electrical and Computer Engineering
EE2003 Engineering Mathematics
Mid-term Test

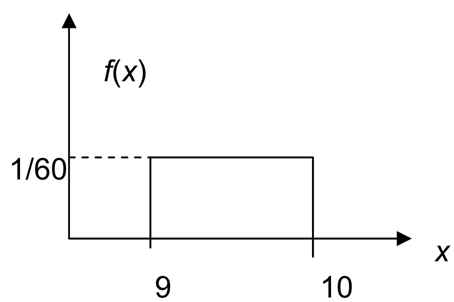
Time Allowed: 50 minutes **Max Marks: 100**

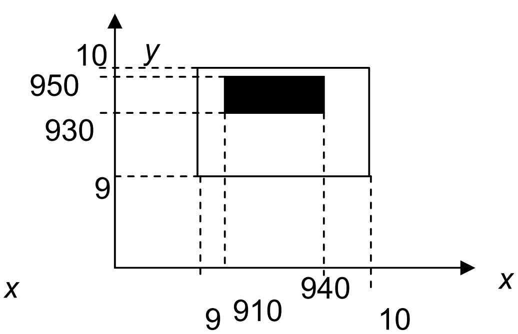
Answer all questions. Mark your answers on the sheet and submit at the end.

1. Two trains arrive independently at a station. For each of the trains, the arrival time is a random variable uniformly distributed between 9 am and 10 am. Let X and Y denote the arrival times for the two trains.
(i) Sketch the pdf of X. Sketch the region **R** over which (X, Y) is defined.
(ii) Mark the region **S** corresponding to the event
A = (9:10 am < $X \leq$ 9:40 am and 9:30 am < $Y \leq$ 9:50 am).
Compute P(A).
(iii) Mark the region **T** corresponding to the event
B = (Only one train is present at the station).
It is known that each train makes a stop of 10 minutes.

(15 marks)

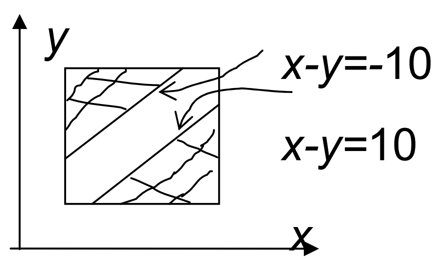
(i)





(ii) The region is marked in the figure on the right. P(A) = P(9:10 am < $X \leq$ 9:40 am and 9:30 am < $Y \leq$ 9:50 am) = P(9:10 am < $X \leq$ 9:40 am) . P(9:30 am < $Y \leq$ 9:50 am) = (30/60)(20/60) = 1/6.

(iii) For trains to not be present at the same time, $|X - Y| > 10$. Thus which ever train comes first, the other does not arrive within 10 minutes, the stop time for the trains. This event is sketched below.



2. For Poisson arrivals, let T be the rv defined as the inter-arrival time measured by the time lapse between two consecutive arrivals. Find the pdf of T.

(12 marks)

For the cdf of T: $P(T \le t)$ = P(at least one arrival in $(0, t)$) = 1 – P(0 arrival in $(0, t)$ = $1 - e^{-\lambda t}$. Therefore, pdf of T is the derivative of cdf given by $\lambda e^{-\lambda t}$, $t \ge 0$.

3. Two mutually exclusive events A and B can be statistically independent.

(a) True (b) False (c) if $P(A) = P(B) = 0.5$ (d) if $P(A) > P(B)$.

ANSWER: (b)

(8 marks)

For mutually exclusive events $P(AB) = 0$ while for statistical independence $P(AB) = P(A).P(B)$. For two expressions to be same, $P(A) = 0$ or $P(B) = 0$ or $P(A) = P(B) = 0$, which is only valid for trivial cases. In general it will not hold. Hence the answer.

4. Someone argues that it is possible to use central limit theorem to show that X in problem 6 is a Gaussian random variable. Show that it is indeed a sound argument.

(12 marks)

Yes it is. X is a sum of 500 statistically independent rv. Note that we assume all throws to be statistically independent. All the conditions for central limit theorem are satisfied.

5. Can X in problem 6 be a Poisson random variable? Justify your answer.

(10 marks)

No as p is not small though N is large. If done so, the approximation will not be a good one.

6. A fair dice is thrown 2,000 times. Define X_i = number of times a multiple of 3 shows up in i-th toss. $P(X = X_1 + X_2 + \ldots + X_{500} = 100)$ is given by

(a) $[500!/(50!50!400!](1/3)^{100}(2/3)^{400}$ (b) $[500!/(50!450!](1/3)^{100}(2/3)^{100}$

(c) $[500!/(100!300!](1/3)^{200}(2/3)^{300}$ (d) $[500!/(100!400!](1/3)^{100}(2/3)^{400}$

ANSWER: (d)

(10 marks)

Multiple of 3 means that either 3 or 6 show up. X_i is either 0 or 1. $P(X_i) = 1$ = P(either 3 or 6 shows up in i-th throw) = 1/3. X is a binomial rv with $N = 500$, $p = 1/3$. Hence $P(X = 100)$ is same as the binomial probability with $N = 500$, $p = 1/3$, $k = 100$.

7. Given a random variable X uniformly distributed over $[-2, 2]$, let $Y = |X|$. $P(Y < 1)$ is

(a) 0.3 (b) 0.4

(c) 0.5 (d) 0.6

ANSWER: (c)

(10 marks)

Y is uniformly distributed over $[0, 2]$. Hence $P(Y < 1) = ½$.

8. Three persons are randomly selected. The probability that they have the same birthday is

(a) close to 0.75 (b) close to 0.5 (c) close to 0.25 (d) close to 0

ANSWER: (d)

(8 marks)

Probability = $\sum_{i=1}^{365} P(X = i, Y = i, Z = i) = \sum_{i=1}^{365} P(X = i)P(Y = i)P(Z = i) = 1/365^2$.

9. Argue how a town having a population of 1 million can be served by a single telephone exchange that can handle only 10,000 calls at any time. Also derive an expression for probability that the exchange is busy when you wish to make a call.

(15 marks)

Assume Bernoulli trials: (1) all users are independent of each other, (2) with prob p, one user wishes to use the system anytime, (3) p is same for all users. The number of users wanting to use the system is binomial with $N = 10^6$ and p. The average value of users in the system is $10^6 p$. As long as $10^6\, p << 10^4$, the system works well. One may also make an approximation of the pdf to Poisson as N is large and p is very small, $\lambda = Np$. In either case, we have

$$P(busy) = P(X > 10{,}000) = \sum_{k=10{,}001}^{1{,}000{,}000} {}^{1{,}000{,}000}C_k\, p^k (1-p)^{1{,}000{,}000-k} \approx e^{-\lambda} \sum_{k=10{,}001}^{1{,}000{,}000} \lambda^k / k!.$$