Hilbert Transform

Given an LTI system having impulse response $h(t)$, the output $y(t)$ for input $x(t)$ is given by

(1) $y(\mathrm{t}) = x(t) \otimes h(t)$.

Here $\otimes$ denotes convolution of two functions and is given by

(2) $$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda \,.$$

Also, if $Y(\omega)$, $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$ and $h(t)$, respectively, then in Fourier domain

(3) $Y(\omega) = X(\omega) \cdot H(\omega)$.

For a very special case, when $h(t) = 1/(\pi t)$, then $y(t)$ is called the **Hilbert transform** of $x(t)$. We now only pursue Hilbert transform in the following. In this case,

(4) $H(\omega) = -\mathrm{j}\ \mathrm{sgn}(\omega)$,

where $\mathrm{sgn}(\omega) = 1$ if $\omega > 0$ and $\mathrm{sgn}(\omega) = -1$ if $\omega < 0$. Thus, Hilbert transform represents a phase shift of $-\pi/2$ for positive frequencies and $\pi/2$ for negative frequencies. No changes in amplitude takes place as $|\ H(\omega)\ | = 1$ for all ω. For the same reason, no change in power spectral density takes place when a random process $\mathbf{X}(\mathrm{t})$ is filtered by this filter. Thus

(5) $S_y(\omega) = S_x(\omega)\ |\ H(\omega)\ |^2 = S_x(\omega)$.

The only thing that remains is to show that

$h(t) = 1/(\pi t)$ has Fourier transform $H(\omega) = -\mathrm{j}\ \mathrm{sgn}(\omega)$.

This is done by noting that if $f(t)$ has Fourier transform $F(\omega)$ then $F(-t)$ has Fourier transform $2\pi\, f(\omega)$. We apply this result with

(6) $f(t) = \mathrm{sgn}(t)$ and $F(\omega) = 2/(j\omega)$.

Proof related to Cauchy's Theorem

Statement:

Let $f(z)$ be an analytic function in domain D. Let z_0 be a point in D and R be the radius of the largest circle with centre z_0 lying inside D. Then there is a power series

(1) $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ which converges to $f(z)$ for $| z - z_0 | < R$. Also

(2) $$c_n = \frac{f^n(z)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz ,$$

where C is any closed path in D which encloses z_0.

Proof. Since $f(z)$ is analytic over D, $f(z) / (z - z_0)^{n+1}$ is analytic everywhere in D except z_0. Hence by Cauchy's theorem, the value of the integral when integrated over a closed path C in D including z_0 does not change when integrated over another path say C^* in D as long as the closed path C^* encloses z_0.

(3) $$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \oint_{C^*} \frac{f(z)}{(z - z_0)^{n+1}} dz .$$

Now we choose the closed path C^* to be a circle with centre z_0 and radius R. The function $f(z)$ is analytic everywhere in this circle as this circle lies inside the region D over which $f(z)$ is given to be analytic. This implies that we can use the power series in (1) to write

(4) $$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \dots + c_n (z - z_0)^n + \dots .$$

It is clear that

(5) $$\frac{d^n}{dz^n} f(z) \bigg|_{z=z_0} = n! c_n$$

Dividing both sides of (4) by $(z - z_0)^{n+1}$ and integrating, we get

(6) $$\oint_{C^*} \frac{f(z)}{(z - z_0)^{n+1}} dz = c_0 \oint_{C^*} \frac{1}{(z - z_0)^{n+1}} dz + c_1 \oint_{C^*} \frac{1}{(z - z_0)^{n}} dz + \dots + c_n \oint_{C^*} \frac{1}{(z - z_0)} dz + \dots$$

Apply the result

(7) $$\oint_{C^*} (z - z_0)^n dz = \begin{cases} 0 & if\ \ n \neq -1 \\ 2\pi i & if\ \ n = -1 \end{cases}$$

to (6) to get

(8) $$\oint_{C^*} \frac{f(z)}{(z - z_0)^{n+1}} dz = c_n \oint_{C^*} \frac{1}{(z - z_0)} dz = 2\pi i \cdot c_n .$$

The original statement of the theorem in (2) is the combination of the statements in (5) and (8).

Note: Please note that we have used both *i* and *j* to denote $\sqrt{-1}$. The notation should be clear from the context and not cause any confusion.